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New Spinor Representation of Maxwell's Equations. II. Algebraic Properties

A. A. Campolattaro[1]

U.E.R. de Physique, Université de Provence, Centre de St. Charles, 13331 Marseille Cedex 03, France

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By using the spinor representation of the electromagnetic field tensor shown in a previous paper (Campolattaro, 1979), several typical algebraic properties of the field are deduced.

1. INTRODUCTION

In the previous paper (Campolattaro, 1980)[2] a new spinor representation of Maxwell's equations was deduced. In this representation for any electromagnetic field, its field tensor $F^{\mu\nu}$ and its dual $*F^{\mu\nu}$ can be written as follows:

$$F^{\mu\nu} = \bar{\Psi} S^{\mu\nu} \Psi \tag{1.1}$$

$$*F^{\mu\nu} = \bar{\Psi} \gamma^5 S^{\mu\nu} \Psi \tag{1.2}$$

where

$$*F^{\mu\nu} = \tfrac{1}{2} \varepsilon^{\mu\nu\sigma\tau} \eta_{\sigma\alpha} \nu_{\tau\beta} F^{\alpha\beta} \tag{1.3}$$

and $\varepsilon^{\mu\nu\sigma\tau}$ the Ricci pseudotensor with entry $+1$ if the parity of the permutation $\mu\nu\sigma\tau$ of the indices $0, 1, 2, 3$ is even, and -1 if odd, and entry zero if two or more indices are equal; $\eta^{\mu\nu}$ is the Minkowski metric tensor

[1]Permanent Address: Physics Department, University of Maryland, Baltimore County Campus, Baltimore, Maryland, U.S.A.
[2]Hereafter referred to as I.

given by

$$\eta^{\mu\nu} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \end{array} \cdot (\mu, \nu = 0-3) \tag{1.4}$$

Moreover Ψ is a spinor, $\overline{\Psi}$ its Dirac conjugate, i.e.,

$$\Psi = \Psi^{\dagger}\gamma^{0} \tag{1.5}$$

with $\Psi^{\dagger}$ the Hermitian conjugate of Ψ, and $S^{\mu\nu}$ the spin operator given by

$$S^{\mu\nu} = \tfrac{1}{2}\gamma^{[\mu}\gamma^{\nu]} \tag{1.6}$$

Here one has

$$\gamma^{[\mu}\gamma^{\nu]} = \tfrac{1}{2}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}) \tag{1.7}$$

with the γ's the Dirac matrices, defined by their anticommutation relations

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu} \tag{1.8}$$

and which in the Dirac representation read

$$\gamma^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}, \quad \gamma^3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{1.9}$$

and γ^5 is given by

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{bmatrix} \tag{1.10}$$

and have the property

$$\gamma^{\mu\dagger} = \gamma^0\gamma^{\mu}\gamma^0 \tag{1.11}$$

Maxwell's equations therefore read as follows[3]:

$$(\overline{\Psi} S^{\mu\nu}\Psi)_{,\mu} = j^{\nu} \tag{1.12}$$

$$(\overline{\Psi}\gamma^5 S^{\mu\nu}\Psi)_{,\mu} = 0 \tag{1.13}$$

Moreover, the duality rotation (Rainich, 1925; Misner and Wheeler, 1957) by the "complexion" α, namely,

$$F^{\mu\nu}\cos\alpha + {}^*F^{\mu\nu}\sin\alpha \tag{1.14}$$

is equivalent to a Touscheck–Nishijima (Touscheck, 1957; Nishijima, 1957) transformation for the spinor Ψ in the spinor Ψ' given by

$$\Psi' = e^{\gamma^5(\alpha/2)}\Psi \tag{1.15}$$

with

$$\cos\alpha = \overline{\Psi}\Psi/\rho \tag{1.16}$$

and

$$\sin\alpha = \overline{\Psi}\gamma^5\Psi/\rho \tag{1.17}$$

where ρ is given by

$$\rho^2 = (\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2 \tag{1.18}$$

In this representation, several algebraic properties of the electromagnetic field are deduced. Several of these properties have been already discussed in the framework of geometry, electromagnetism, and gravitation by several authors (Rainich, 1925; Misner and Wheeler, 1957; Ruse, 1936; Whitten, 1959, 1962) and are represented here, in the above-mentioned spinor formalism, for their applications in forthcoming papers.

2. ALGEBRAIC PROPERTIES OF THE ELECTROMAGNETIC FIELD

In the above-mentioned representation the two invariants I_1 and I_2 given by

$$I_1 = \tfrac{1}{2}F_{\mu\nu}F^{\mu\nu} = H^2 - E^2 \tag{2.1}$$

[3]Hereafter we shall use the Einstein sum convention under which the sum is understood when two indices are repeated, and a comma followed by an index indicates the operation of partial derivative with respect to the variable with that index.

and

$$I_2 = \tfrac{1}{2} F_{\mu\nu} {}^*F^{\mu\nu} = 2\mathbf{E}\cdot\mathbf{H} \tag{2.2}$$

with **H** and **E**, respectively, the magnetic and electric field, become, by means of (A.14) and (A.16) from Appendix A,

$$I_1 = \tfrac{1}{4}\left[(\overline{\Psi}\Psi)^2 - (\overline{\Psi}\gamma^5\Psi)^2 \right] \tag{2.3}$$

$$I_2 = \tfrac{1}{2}(\overline{\Psi}\Psi)(\overline{\Psi}\gamma^5\Psi) \tag{2.4}$$

Therefore

$$I_1^2 + I_2^2 = \frac{1}{16}\rho^4 \tag{2.5}$$

By subtracting from (A.11), (A.12) and by using (A.14), one has

$$\begin{aligned}&(\overline{\Psi}S^{\mu\nu}\Psi)(\overline{\Psi}S_{\mu\sigma}\Psi) - (\overline{\Psi}\gamma^5S^{\mu\nu}\Psi)(\overline{\Psi}\gamma^5S_{\mu\sigma}\Psi)\\ &\quad = \tfrac{1}{2}(\overline{\Psi}S^{\mu\nu}\Psi)(\overline{\Psi}S_{\mu\nu}\Psi)\delta_\sigma^\nu = \tfrac{1}{4}\left\{ (\overline{\Psi}\Psi)^2 - (\overline{\Psi}\gamma^5\Psi)^2 \right\}\delta_\sigma^\nu\end{aligned} \tag{2.6}$$

which is the spinor equivalent of the identity

$$F^{\mu\nu}F_{\mu\sigma} - {}^*F^{\mu\nu}{}^*F_{\mu\sigma} = \tfrac{1}{2}F^{\alpha\beta}F_{\alpha\beta}\delta_\sigma^\nu \tag{2.7}$$

On the other hand the other well-known identity

$$F_{\mu\sigma}{}^*F^{\mu\nu} = \tfrac{1}{4}F_{\alpha\beta}{}^*F^{\alpha\beta}\delta_\sigma^\mu \tag{2.8}$$

is nothing but the identity (B.9)

$$(\overline{\Psi}S_{\mu\sigma}\Psi)(\overline{\Psi}\gamma^5S^{\mu\nu}\Psi) = \tfrac{1}{4}(\overline{\Psi}\Psi)(\overline{\Psi}\gamma^4\Psi)\delta_\sigma^\nu$$

once (A.16) is used. Moreover, by using the usual definition for the energy-momentum tensor, namely,

$$T_\nu^\mu = F^{\mu\sigma}F_{\nu\sigma} - \tfrac{1}{4}\delta_\nu^\mu F^{\sigma\rho}F_{\sigma\rho} \tag{2.9}$$

one has for (A.11) and (A.14)

$$T_\nu^\mu = \tfrac{1}{8}\left[\rho^2\delta_\nu^\mu - 2(\overline{\Psi}\gamma^\mu\Psi)(\overline{\Psi}\gamma_\nu\Psi) - 2(\overline{\Psi}\gamma^5\gamma^\mu\Psi)(\overline{\Psi}\gamma^5\gamma_\nu\Psi) \right] \tag{2.10}$$

which by means of (A.2), (A.3), and (A.4), readily gives

$$T^{\mu}_{\nu}T^{\nu}_{\sigma}=\tfrac{1}{64}\rho^4\delta^{\mu}_{\sigma} \tag{2.11}$$

from which the other

$$T^{\mu}_{\nu}T^{\nu}_{\mu}=\tfrac{1}{16}\rho^4 \tag{2.12}$$

The utilization of the identities (A.2), (A.3), and (A.4) permits one immediately to have from (2.3) the following:

$$T^{\mu}_{\nu}(\overline{\Psi}\gamma^{\nu}\Psi)=-\tfrac{1}{8}\rho^2(\overline{\Psi}\gamma^{\mu}\Psi) \tag{2.13}$$

and

$$T^{\mu}_{\nu}(\overline{\Psi}\gamma^5\gamma^{\nu}\Psi)=-\tfrac{1}{8}\rho^2(\overline{\Psi}\gamma^5\gamma^{\mu}\Psi) \tag{2.14}$$

By means of these two vectors we can construct two complex vectors k_{μ} and l_{μ}, which are both null, i.e., $k_{\mu}k^{\mu}=0$ and $l_{\mu}l^{\mu}=0$ and such that $k_{\mu}l^{\mu}=1$. They are

$$k_{\mu}=\frac{1}{2^{1/2}\rho}\left[(\overline{\Psi}\gamma_{\mu}\Psi)+i(\overline{\Psi}\gamma^5\gamma_{\mu}\Psi)\right] \tag{2.15}$$

and

$$l_{\mu}=\frac{1}{2^{1/2}\rho}\left[(\overline{\Psi}\gamma_{\mu}\Psi)-i(\overline{\Psi}\gamma^5\gamma_{\mu}\Psi)\right] \tag{2.16}$$

which, by means of (2.13) and (2.14), are still eigenvectors of T^{μ}_{ν} corresponding to the same eigenvalue $-\frac{1}{8}\rho^2$, i.e.,

$$T^{\mu}k^{\nu}=-\tfrac{1}{8}\rho^2k^{\mu} \tag{2.17}$$

$$T^{\mu}_{\nu}l^{\nu}=-\tfrac{1}{8}\rho^2l^{\nu} \tag{2.18}$$

It is readily seen that the two complex vectors k_{μ} and l_{μ} are also eigenvectors of F^{μ}_{ν} and $*F^{\mu}_{\nu}$ corresponding, respectively, to the eigenvalues $\pm\frac{1}{2}\rho\sin\alpha$ and $\mp\frac{1}{2}\rho\cos\alpha$. This result is quite trivial once one takes the four identities (A.6), (A.8), (A.9), and (A.10) and remembers (2.15) and (2.16)

$$(\bar{\Psi}S^{\mu\nu}\Psi)k_\nu = -\tfrac{1}{2}\rho(\sin\alpha)k^\mu \tag{2.19}$$

$$(\bar{\Psi}S^{\mu\nu}\Psi)l_\nu = \tfrac{1}{2}\rho(\sin\alpha)l^\mu \tag{2.20}$$

$$(\bar{\Psi}\gamma^5 S^{\mu\nu}\Psi)k_\nu = \tfrac{1}{2}\rho(\cos\alpha)k^\mu \tag{2.21}$$

$$(\bar{\Psi}\gamma^5 S^{\mu\nu}\Psi)l_\nu = -\tfrac{1}{2}\rho(\cos\alpha)l^\mu \tag{2.22}$$

with $\cos\alpha$ and $\sin\alpha$ given by (2.11) and (2.12).

Moreover one has by straightforward calculations

$$k^\mu l_\nu + k_\nu l^\mu = \frac{1}{\rho^2}\left[(\bar{\Psi}\gamma^\mu\Psi)(\bar{\Psi}\gamma_\nu\Psi) + (\bar{\Psi}\gamma^5\gamma^\mu\Psi)(\bar{\Psi}\gamma^5\gamma_\nu\Psi)\right] \tag{2.23}$$

By substituting into the expression for T_ν^μ given by (2.10) one has the representation of the energy-momentum tensor in terms of these vectors k_μ and l_μ, i.e.,

$$T_\nu^\mu = \frac{\rho^2}{8}\left[(\delta_\nu^\mu - 2(k^\mu l_\nu + k_\nu l^\mu)\right] \tag{2.24}$$

The electromagnetic field tensor as well as its dual are both susceptible of being represented in terms of their common eigenvectors, k_μ and l_μ, as has been done in the previous section for the energy-momentum tensor. In fact by taking (2.15) and (2.16), by means of (A.2), (A.3), and (A.4) one readily has

$$k^{[\mu}l^{\nu]} = \frac{1}{2}(k^\mu l^\nu - k^\nu l^\mu) = \frac{1}{2\rho^2}\left[(\bar{\Psi}\gamma^5\gamma^\mu\Psi)(\bar{\Psi}\gamma^\nu\Psi) - (\bar{\Psi}\gamma^5\gamma^\nu\Psi)(\bar{\Psi}\gamma^\mu\Psi)\right] \tag{2.25}$$

which by using (B.11) reads

$$k^{[\mu}l^{\nu]} = \frac{1}{\rho^2}\left[(\bar{\Psi}\gamma^5\Psi)(\bar{\Psi}S^{\nu\mu}\Psi) - (\bar{\Psi}\Psi)(\bar{\Psi}\gamma^5 S^{\nu\mu}\Psi)\right] \tag{2.26}$$

On the other hand (1.16) and (1.17) hold and (2.26) can be written as follows:

$$k^{[\mu}l^{\nu]} = \frac{1}{\rho}\left[(\bar{\Psi}S^{\nu\mu}\Psi)\sin\alpha - (\bar{\Psi}\gamma^5 S^{\nu\mu}\Psi)\cos\alpha\right] \tag{2.27}$$

Equation (2.27) states that the bivector $\rho k^{[\mu}l^{\nu]}$ coincides with the tensor $\bar{\Psi}S^{\nu\mu}\Psi$ after a duality rotation by an angle $\alpha-\pi/2$. Therefore the tensor $\bar{\Psi}S^{\nu\mu}\Psi$ will be the bivector $\rho k^{[\mu}l^{\nu]}$ after a duality rotation by an angle $\pi/2-\alpha$, i.e.,

$$(\bar{\Psi}S^{\nu\mu}\Psi)=\rho\left[k^{[\mu}l^{\nu]}\sin\alpha+{}^*(k^{[\mu}l^{\nu]})\cos\alpha\right] \tag{2.28}$$

From equation (2.28) one readily finds the corresponding equation for the dual of $(\bar{\Psi}S^{\mu\nu}\Psi)$. In fact by taking the dual on both sides of (2.28) one has

$$\left(\bar{\Psi}\gamma^5S^{\mu\nu}\Psi\right)=\rho\left[k^{[\mu}l^{\nu]}\cos\alpha-{}^*k^{[\mu}l^{\nu]}\sin\alpha\right] \tag{2.29}$$

3. CONCLUSIONS

The typical algebraic properties of the electromagnetic field tensor have been deduced in the spinor formalism after the spinor representation given in I.

In the two appendices, two mathematical techniques for obtaining several spinor identities have been exposed and utilized for deducing those that have been used in the text.

APPENDIX A: SOME IDENTITIES THAT HAVE BEEN USED IN THE TEXT

In I, it was shown that for any three spinors Φ_1, Φ_2, and Ψ the following fundamental identity holds:

$$\left(\bar{\Phi}_1\gamma^\mu\Psi\right)\left(\bar{\Phi}_2\gamma_\mu\Psi\right)=\left(\bar{\Phi}_1\Psi\right)\left(\bar{\Phi}_2\Psi\right)+\left(\bar{\Phi}_1\gamma^5\Psi\right)\left(\bar{\Phi}_2\gamma^5\Psi\right) \tag{A.1}$$

From this many other useful identities are deduced. By taking

$$\Phi_1=\Psi \qquad \text{and} \qquad \Phi_2=\Psi$$

one readily has

$$\left(\bar{\Psi}\gamma^\mu\Psi\right)\left(\bar{\Psi}\gamma_\mu\Psi\right)=\left(\bar{\Psi}\Psi\right)^2+\left(\bar{\Psi}\gamma^5\Psi\right)^2=\rho^2 \tag{A.2}$$

with

$$\Phi_1=\gamma^5\Psi \qquad \text{and} \qquad \Phi_2=\gamma^5\Psi$$

one has

$$\left(\overline{\Psi}\gamma^5\gamma^\mu\Psi\right)\left(\overline{\Psi}\gamma^5\gamma_\mu\Psi\right)=(\overline{\Psi}\Psi)^2+\left(\overline{\Psi}\gamma^5\Psi\right)^2=\rho^2 \tag{A.3}$$

and by taking

$$\Phi_1=\Psi \qquad \text{and} \qquad \Phi_2=\gamma^5\Psi$$

one has

$$\left(\overline{\Psi}\gamma^\mu\Psi\right)\left(\overline{\Psi}\gamma^5\gamma_\mu\Psi\right)=0 \tag{A.4}$$

Another important identity is the following:

$$\left(\overline{\Psi}\gamma^\mu\gamma^\nu\Psi\right)\left(\overline{\Psi}\gamma_\nu\Psi\right)=\left(\overline{\Psi}\gamma^\mu\Psi\right)(\overline{\Psi}\Psi)+\left(\overline{\Psi}\gamma^\mu\gamma^5\Psi\right)\left(\overline{\Psi}\gamma^5\Psi\right) \tag{A.5}$$

whose demonstration is readily achieved by taking into (A.1)

$$\Phi_1=\gamma^\mu\Psi \qquad \text{and} \qquad \Phi_2=\Psi$$

and by noticing that since one has (1.11) one also has

$$\overline{\Phi}_1=\overline{\Psi}\gamma^\mu$$

By means of (A.5) the following identity is readily found:

$$\left(\overline{\Psi}S^{\mu\nu}\Psi\right)\left(\overline{\Psi}\gamma_\nu\Psi\right)=-\tfrac{1}{2}i\left(\overline{\Psi}\gamma^5\Psi\right)\left(\overline{\Psi}\gamma^5\gamma^\mu\Psi\right) \tag{A.6}$$

In fact by using (1.6), (1.7), and (1.8) one has

$$S^{\mu\nu}=\tfrac{1}{2}i(\gamma^\mu\gamma^\nu-\eta^{\mu\nu})=\tfrac{1}{2}i(\eta^{\mu\nu}-\gamma^\nu\gamma^\mu) \tag{A.7}$$

and by means of (A.5) one obtains (A.6). The same technique applies for the other identities

$$\left(\overline{\Psi}S^{\mu\nu}\Psi\right)\left(\overline{\Psi}\gamma^5\gamma_\nu\Psi\right)=\tfrac{1}{2}i\left(\overline{\Psi}\gamma^5\Psi\right)\left(\overline{\Psi}\gamma^\mu\Psi\right) \tag{A.8}$$

obtained from (A.1) with the position

$$\Phi_1=\gamma^\mu\Psi \qquad \text{and} \qquad \Phi_2=\gamma^5\Psi$$

$$\left(\overline{\Psi}\gamma^5S^{\mu\nu}\Psi\right)\left(\overline{\Psi}\gamma_\nu\Psi\right)=\tfrac{1}{2}i(\overline{\Psi}\Psi)\left(\overline{\Psi}\gamma^5\gamma^\mu\Psi\right) \tag{A.9}$$

with

$$\Phi_1=\gamma^{\mu}\gamma^5\Psi \qquad \text{and} \qquad \Phi_2=\Psi$$

and

$$(\bar{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\bar{\Psi}\gamma^5\gamma_\nu\Psi)=-\tfrac{1}{2}i(\bar{\Psi}\Psi)(\bar{\Psi}\gamma^{\mu}\Psi) \tag{A.10}$$

with

$$\Phi_1=\gamma^{\mu}\gamma^5\Psi \qquad \text{and} \qquad \Phi_2=\gamma^5\Psi$$

Similarly one has the other identity

$$(\bar{\Psi}S^{\mu\nu}\Psi)(\bar{\Psi}S_{\mu\sigma}\Psi)=\tfrac{1}{4}\Big[(\bar{\Psi}\Psi)^2\delta_\sigma^\nu-(\bar{\Psi}\gamma^\nu\Psi)(\bar{\Psi}\gamma_\sigma\Psi)$$
$$-(\bar{\Psi}\gamma^5\gamma^\nu\Psi)(\bar{\Psi}\gamma^5\gamma_\sigma\Psi)\Big] \tag{A.11}$$

$$(\bar{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\bar{\Psi}\gamma^5 S_{\mu\sigma}\Psi)=\tfrac{1}{4}\Big[(\bar{\Psi}\gamma^5\Psi)^2\delta_\sigma^\nu-(\bar{\Psi}\gamma^\nu\Psi)(\bar{\Psi}\gamma_\sigma\Psi)$$
$$-(\bar{\Psi}\gamma^5\gamma^\nu\Psi)(\bar{\Psi}\gamma^5\gamma_\sigma\Psi)\Big] \tag{A.12}$$

$$(\bar{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\bar{\Psi}S_{\mu\sigma}\Psi)=\tfrac{1}{4}\Big[(\bar{\Psi}\Psi)(\bar{\Psi}\gamma^5\Psi)\delta_\sigma^\nu+(\bar{\Psi}\Psi)(\bar{\Psi}\gamma^5\gamma^\nu\gamma_\sigma\Psi)$$
$$-(\bar{\Psi}\gamma^5\Psi)(\bar{\Psi}\gamma^\nu\gamma_\sigma\Psi)+(\bar{\Psi}\gamma^\nu\Psi)(\bar{\Psi}\gamma^5\gamma_\sigma\Psi)-(\bar{\Psi}\gamma^5\gamma^\nu\Psi)(\bar{\Psi}\gamma_\sigma\Psi)\Big] \tag{A.13}$$

And from (A.11), (A.12), and (A.13) one readily has by contraction

$$(\bar{\Psi}S^{\mu\nu}\Psi)(\bar{\Psi}S_{\mu\nu}\Psi)=\tfrac{1}{2}\Big[(\bar{\Psi}\Psi)^2-(\Psi\gamma^5\Psi)^2\Big] \tag{A.14}$$

$$(\bar{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\bar{\Psi}\gamma^5 S_{\mu\nu}\Psi)=-\tfrac{1}{2}\Big[(\bar{\Psi}\Psi)^2-(\bar{\Psi}\gamma^5\Psi)^2\Big] \tag{A.15}$$

and

$$(\bar{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\bar{\Psi}S_{\mu\nu}\Psi)=(\bar{\Psi}\Psi)(\bar{\Psi}\gamma^5\Psi) \tag{A.16}$$

APPENDIX B: A TECHNIQUE FOR DETERMINING OTHER SPINOR IDENTITIES

A simple technique for deducing many of these identities is based on the use of duality rotation transformation of spinors (1.15). An example will be enough for clarifying the technique. The identity (A.14) is true for any spinor Ψ, hence it is true also for the spinor Ψ', which is obtained, in particular, from the spinor Ψ by means of a duality transformation of an angle $\pi/4$ so that (A.14) reads

$$\begin{aligned}&(\overline{\Psi} e^{\gamma^5\pi/4} S^{\mu\nu} e^{\gamma^5\pi/4}\Psi)(\overline{\Psi} e^{\gamma^5\pi/4} S_{\mu\nu} e^{\gamma^5\pi/4}\Psi)\\&=\tfrac{1}{2}\Big[(\overline{\Psi} e^{\gamma^5\pi/2}\Psi)^2-(\overline{\Psi} e^{\gamma^5\pi/4}\gamma^5 e^{\gamma^5\pi/4}\Psi)^2\Big]\end{aligned}$$

which by using the anticommutation properties of the γ;s, gives (A.15). An application of this technique will unable us to demonstrate straightforwardly some other identities.

Our objective is to demonstrate the following:

$$(\overline{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\overline{\Psi} S_{\mu\sigma}\Psi)=(\overline{\Psi} S^{\mu\nu}\Psi)(\overline{\Psi}\gamma^5 S_{\mu\sigma}\Psi) \tag{B.1}$$

By indicating with a subscript the angle by which the spinor is dually rotated, we have

$$(\overline{\Psi} S_{\mu\sigma}\Psi)_{\pi/8}=2^{-1/2}\big[(\overline{\Psi} S_{\mu\sigma}\Psi)+(\overline{\Psi}\gamma^5 S_{\mu\sigma}\Psi)\big] \tag{B.2}$$

and

$$(\overline{\Psi}\gamma^5 S_{\mu\sigma}\Psi)_{\pi/8}=2^{-1/2}\big[(\overline{\Psi}\gamma^5 S_{\mu\sigma}\Psi)-(\overline{\Psi} S_{\mu\sigma}\Psi)\big] \tag{B.3}$$

so that one has

$$\begin{aligned}\big[(\overline{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\overline{\Psi} S_{\mu\sigma}\Psi)\big]_{\pi/8}&=\tfrac{1}{2}\big[(\overline{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\overline{\Psi}\gamma^5 S_{\mu\sigma}\Psi)-(\overline{\Psi} S^{\mu\nu}\Psi)(\overline{\Psi} S_{\mu\sigma}\Psi)\\&\quad+(\overline{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\overline{\Psi} S_{\mu\sigma}\Psi)-(\overline{\Psi} S^{\mu\nu}\Psi)(\overline{\Psi}\gamma^5 S_{\mu\sigma}\Psi)\big]\end{aligned} \tag{B.4}$$

which by means of (A.11) and (A.12) gives

$$\begin{aligned}\big[(\overline{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\Psi S_{\mu\sigma}\Psi)\big]_{\pi/8}&=\tfrac{1}{2}\Big\{-\tfrac{1}{4}\Big[(\overline{\Psi}\Psi)^2-(\overline{\Psi}\gamma^5\Psi)^2\Big]\delta^{\nu}_{\sigma}\\&\quad+(\overline{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\overline{\Psi} S_{\mu\sigma}\Psi)-(\overline{\Psi} S^{\mu\nu}\Psi)(\Psi\gamma^5 S_{\mu\sigma}\Psi)\Big\}\end{aligned} \tag{B.5}$$

By a duality rotation of the spinor Ψ by an angle $\pi/4$, since the product of two duality rotations is commutative, (B.5) reads

$$\left[(\bar{\Psi}S^{\mu\nu}\Psi)(\bar{\Psi}\gamma^5 S_{\mu\sigma}\Psi)\right]_{\pi/8}=\tfrac{1}{2}\left\{\tfrac{1}{4}\left[(\bar{\Psi}\Psi)^2-(\bar{\Psi}\gamma^5\Psi)^2\right]\right.$$
$$\left.+(\bar{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\bar{\Psi}S_{\mu\sigma}\Psi)-(\bar{\Psi}S^{\mu\nu}\Psi)(\bar{\Psi}\gamma^5 S_{\mu\sigma}\Psi)\right\} \tag{B.6}$$

The comparison of (B.5) and (B.6) gives

$$\left[(\bar{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\bar{\Psi}S_{\mu\sigma}\Psi)\right]_{\pi/8}=\left[(\bar{\Psi}S^{\mu\nu}\Psi)(\bar{\Psi}\gamma^5 S_{\mu\sigma}\Psi)\right]_{\pi/8} \tag{B.7}$$

A further duality rotation by $-\pi/8$ permits one to demonstrate the identity (A.17). Equation (B.7) then reads

$$\left[(\bar{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\bar{\Psi}S_{\mu\sigma}\Psi)\right]_{\pi/8}=-\tfrac{1}{8}\left[(\bar{\Psi}\Psi)^2-(\bar{\Psi}\gamma^5\Psi)^2\right]\delta_\sigma^\nu \tag{B.8}$$

which, by a duality rotation by an angle $-\pi/8$ gives

$$(\bar{\Psi}\gamma^5 S^{\mu\nu}\Psi)(\bar{\Psi}S_{\mu\sigma}\Psi)=\tfrac{1}{4}(\bar{\Psi}\Psi)(\bar{\Psi}\gamma^5\Psi)\delta_\sigma^\nu \tag{B.9}$$

By substituting the last result into (A.13) one has

$$(\bar{\Psi}\Psi)(\bar{\Psi}\gamma^5\gamma^\nu\gamma_\sigma\Psi)-(\bar{\Psi}\gamma^5\Psi)(\bar{\Psi}\gamma^\nu\gamma_\sigma\Psi)$$
$$=(\bar{\Psi}\gamma_\sigma\Psi)(\bar{\Psi}\gamma^5\gamma^\nu\Psi)-(\bar{\Psi}\gamma^\nu\Psi)(\bar{\Psi}\gamma^5\gamma_\sigma\Psi) \tag{B.10}$$

which written in contravariant form and by using (A.7) reads

$$(\bar{\Psi}\gamma^5\Psi)(\bar{\Psi}S^{\nu\sigma}\Psi)-(\bar{\Psi}\Psi)(\bar{\Psi}\gamma^5 S^{\nu\sigma}\Psi)$$
$$=\tfrac{1}{2}i\left[(\bar{\Psi}\gamma^\nu\Psi)(\bar{\Psi}\gamma^5\gamma^\sigma\Psi)-(\bar{\Psi}\gamma^{5\gamma}\Psi)(\Psi\gamma^\sigma\Psi)\right] \tag{B.11}$$

REFERENCES

Campolattaro, A. A. (1980). *International Journal of Theoretical Physics*, **19**, 99.
Misner, C. W., and Wheeler, J. A. (1957). *Annals of Physics*, **2**, 525.
Nishijima, K. (1957). *Nuovo Cimento*, **5**, 1349.

Rainich, G. Y. (1925). *Transactions of the American Mathematical Society*, **27**, 106.
Ruse, H. S. (1936). *Proceedings of the London Mathematical Society* **41**, 302.
Touschek, B. F. (1957). *Nuovo Cimento*, **5**, 1281.
Witten, L. (1959). *Physical Review*, **115**, 206.
Witten, L. (1962). *Gravitation, An Introduction to Current Research*, p. 382, L. Witten, ed. John Wiley and Sons, New York.